Unitarity restrictions

We seek irreducible unitary modules in the diagonal compact subgroup R-sym $\times SO(2) \times SO(d)$

→ states are labeled by:

- R-sym weights
- scaling dimension (Eigenvalue of D')
- Lorentz Group weights.

→ seek $SO(d) \times SO(2) \times R$ rep with <u>lowest</u> D' eigenvalue Δ_0

superconformal states are obtained from the conformal ones by action of Q'-operator:

$$Q'_{iu} \, | \{s\} m \rangle$$

SO(n) R-sym weights   Lorentz SO(d) weights

since $Q'^{\dagger} = S'$ we have

$$A_{v\{t\}m'j,\, u\{s\}mi} = \langle \{t\} m' | S'_{jv} \, Q'_{iu} | \{s\} m \rangle$$

$$\geq 0$$

has to be positive!

$\underline{d=3}$:

states : $Q'_{im} \, |\{s\} m \rangle$

SO(n) R-sym weights   SO(3) Lorentz sym weights

$\Rightarrow$ require $\langle \{t\} m' | S'_{j\nu} Q'_{im} | \{s\} m \rangle \geq 0$

Using $D' = -i \Delta_0$ and

$$\{ Q'_{i\alpha}, S'_{j\tilde{\beta}} \} = i \frac{\delta_{ij}}{2} \left[(M'_{\mu\nu} \Gamma_\mu \Gamma_\nu C)_{\alpha\tilde{\beta}} + 2D' \delta_{\alpha\beta} \right]$$
$$- (i) \, \delta_{\alpha\tilde{\beta}} \, I_{ij}$$

we see that negative eigenvalues of matrix B must have modulus $\leq \Delta_0$ where

$$B_{\nu\{t\}m'j,\,\mu\{s\}mi} = \left(i \frac{\delta_{ij}}{2}\right) \left((M_{ab})_{m'm} \Gamma_a \Gamma_b \right)_{\mu\nu}$$
$$- (i) \, \delta_{\mu\nu} \, (I_{ij})_{\{s\}\{t\}}$$

sum of two matrices :

- one <u>only</u> with Lorentz indices
- one <u>only</u> with R-sym indices

$\rightarrow$ diagonalize pieces separately
add eigenvalues

1) R-sym piece to diagonalization of M_{ij} in the case of CA.

$$\rightarrow \quad -\frac{1}{2}\left[C_2(R') + C_2(R) + C_2(\text{vector})\right] \qquad (*)$$

$R \times \text{vector} = \cdots + R' + \cdots$

pick R' with largest Casimir

For R with weights $(h_1, h_2, \cdots, h_n)$

$\rightarrow R'$ with largest Casimir: $(h_1+1, h_2, \cdots, h_n)$

Using $C_2(\{h\}) = \sum_{i=1}^{n} (h_i^2 + (2n-2i)h_i)$

we get: most negative eigenvalue of $(*)$
$= -h_1$

2) Lorentz part is diagonalized similarly

$\rightarrow$ diagonalize operator $2 \cdot J \cdot S$

(recall $S = \frac{i}{4}[T_a^1, T_b^1]$)

where S is spin operator and J angular momentum operator in given rep.

$\rightarrow$ lowest eigenvalues: 0 for $j=0$
$-(j+1)$ otherwise

$$\Rightarrow \quad \Delta_0 \geq h_1 \quad (j=0)$$
$$\Delta_0 \geq h_1 + j + 1 \quad (j \neq 0)$$

$\underline{d=4:}$

Demand positivity of

$$A_{\nu\{t\}m'n'j,\mu\{s\}mni} = \langle \{t\}m'n' | S'_{j\nu}\, Q'_{i\mu} | \{s\}mn \rangle$$

U(n) weights

J_1 and J_2 of SO(4) Lorentz

Let j_1 and j_2 be highest weights of the J_1- and J_2-representations respectively.

Use

$$\{Q'_{i\alpha}, S'_{j\tilde{\beta}}\} = \left((i)\delta_{ij}/2\right)\left[(M'_{\mu\nu}\Gamma_\mu\Gamma_\nu)_{\alpha\tilde{\beta}} + 2\delta_{\alpha\tilde{\beta}}D'\right]$$
$$-2(P_+)_{\alpha\tilde{\beta}}T_{ij} + 2(P_-)_{\alpha\tilde{\beta}}T_{ji} + \tfrac{1}{2}(\sigma_5)_{\alpha\tilde{\beta}}R$$

$\rightarrow$ condition for unitary representations:

$$\Delta_0 \geq -(\text{smallest eigenvalue of } \mathcal{B})$$

where

$$\mathcal{B} = (M'_{ab}\Gamma_a\Gamma_b)_{\mu\nu} - 2(P_+)_{\mu\nu}T_{ij}$$
$$+2(P_-)_{\mu\nu}T_{ji} + \tfrac{1}{2}(\sigma_5)_{\mu\nu}R$$

gen. of U(1) subalgebra of U(n)

$\rightarrow$ is direct sum of + chirality and − chirality parts:

$$P_+ = 4J_1\cdot S_1 - 2T_{ij}V_{ij} + R/2$$
$$P_- = 4J_2\cdot S_2 - 2T_{ij}(V^*)_{ij} + \tfrac{1}{2}R$$

where V_{ij} is the fundamental SU(n) matrix generator and V^* the anti-fundamental one

Use $T \cdot V = (T+V)^2 - T^2 - V^2$

$\rightarrow C(T \times V) - C(T) - C(V)$

eigenvalues of T: $\{r_i\}$ (number of boxes in Young tableaux in ith row)

$\rightarrow$ $T \times V$ can contain only reps with

$r_K' = r_K + 1$ for some K (adding box at Kth row)

as long as $R_K = r_K - r_{K-1} < 0$

$$\rightarrow \left(2T \cdot V - \tfrac{1}{2} R\right) = \left(2r_K - \frac{2\sum_i' r_i}{n} + 2 - 2K + \frac{R(4-n)}{2n}\right)$$

Coupling to V^*:

$r_i' = r_i - 1$ for $i = K$ (removing a box at Kth row)

only when $R_{K+1} = r_{K+1} - r_K < 0$

$$\rightarrow \left(2T \cdot V^* - R/2\right) = \left(-2r_K + \frac{2\sum_i' r_i}{n} - 2n + 2K - \frac{R(4-n)}{2n}\right)$$

$2J \cdot S$ has usual values $-(j+1)$ and j.

$\rightarrow$ we get zero total eigenvalues when:

$$\Delta_0 = d^1_{nK} = 2j_1 + 2 - 2r_K + \frac{2\sum_i' r_i}{n} + 2K - \frac{R(4-n)}{2n} - 2n + (-2\delta_{j,0})$$

$$\Delta_0 = d^2_{nK} = 2j_1 - 2r_K + \frac{2\sum_i' r_i}{n} + 2K - \frac{R(4-n)}{2n} - 2n$$

where K runs over values $R_K \neq 0$.

For K running over values with $R_{K+1} \neq 0$, we get

$$\Delta_0 = d^3_{nK} = 2j_2 + 2 - 2r_K + \frac{2\sum_i r_i}{n} + 2K - \frac{R(4-n)}{2n} - 2n + (-2\delta_{0\varepsilon})$$

$$\Delta_0 = d^4_{nK} = 2j_2 - 2r_K + \frac{2\sum_i r_i}{n} + 2K - \frac{R(4-n)}{2n} - 2n$$

We have to require: $\Delta_0 \geq \max(d^1, d^2, d^3, d^4)$

Since $d^1 \geq d^2$ and $d^3 \geq d^4$ this reduces to

$$\Delta_0 \geq \max(d^1_{n1}, d^3_{nn})$$

as d^1 takes its maximum at $K=1$

and d^3 takes its maximum at $K=n$.

<u>$d=5$:</u>

highest weights of $SO(5)$: (h_1, h_2)

" " " R-symmetry group $SO(3)$: K

$$\longrightarrow \Delta_0 \geq -\left(C_2(R') - C_2(R) - \tfrac{5}{2}\right) + 3K$$

$R \times$ spinor $= \dots + R' + \dots$ $SO(5)$-re

Values for zeros of our "A" matrix:

$$\Delta_0 = -\left(C_2(R') - C_2(R) - \tfrac{5}{2}\right) + 3K$$

$$\Delta_0 = -\left(C_2(R') - C_2(R) - \tfrac{5}{2}\right) - 3K - 3(1 - \delta_{K0})$$

$\underline{d=6:}$

$SO(6)$ highest weights: (h_1, h_2, h_3)

R-symmetry groups: $SO(3)$ $(n=1)$ h.w.: K

$SO(5)$ $(n=2)$ h.w.: (l_1, l_2)

i) $\underline{n=1}$:

$$\Delta_0 \geq -\left(C_2(R') - C_2(R) - \frac{15}{4}\right) + 4K$$

where R is $SO(5)$-rep $\{h_i\}$

R' appears in $R \times$ chiral spinor $(SO(6)) = \cdots R' + \cdots$

Values of zeros of "A" matrix:

$$\Delta_0 = -\left(C_2(R') - C_2(R) - \frac{15}{4}\right) + 4K$$

$$\Delta_0 = -\left(C_2(R') - C_2(R) - \frac{15}{4}\right) - 4K - 4 + 4\delta_{K0}$$

For example, we can have the following combinations for $-\left(C_2(R') - C_2(R) - \frac{15}{4}\right)$:

$$-h_1 - h_2 - h_3$$

$$-h_1 + h_2 + h_3 + 2 \qquad \text{if } h_2 - \tfrac{1}{2} \geq |h_3 - \tfrac{1}{2}|$$

$$+h_1 - h_2 + h_3 + 4 \qquad \text{if } h_1 \neq h_2$$

$$+h_1 + h_2 - h_3 + 6 \qquad \text{if } h_2 - \tfrac{1}{2} \geq |h_3 + \tfrac{1}{2}|$$

$\underline{n=2}$:

$$\Delta_0 \geq -\left(C_2(R') - C_2(R) - \frac{15}{4}\right) + 2(l_1 + l_2)$$

similar analysis as above